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RESOLUTION PROPERTIES OF THE FOURIER METHOD FOR DISCONTINUOUS WAVES [1]

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ABSTRACT

In this paper we discuss the wave-resolution properties of the Fourier approximations of a wave function with discontinuities. It is well known that a minimum of two points per wave is needed to resolve a periodic wave function using Fourier expansions. For Chebyshev approximations of a wave function, a minimum of π points per wave is needed [3]. Here we obtain an estimate for the minimum number of points per wave to resolve a discontinuous wave based on its Fourier coefficients.

In our recent work on overcoming the Gibbs phenomenon, we have shown that the Fourier coefficients of a discontinuous function contain enough information to reconstruct with exponential accuracy the coefficients of a rapidly converging Gegenbauer expansion. We therefore study the resolution properties of a Gegenbauer expansion where both the number of terms and the order increase.

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1 Introduction

In [5] (see also [3]), the issue of the number of points required to resolve a wave has been considered as a measure of the accuracy of a given scheme. In particular it has been shown that only two points per wave are required if the wave function is approximated by a Fourier series. This result is almost trivial since the expansion basis contains the wave function. It is also shown in [3] that if the wave function $f(x) = e^{i\pi\omega x}$ is approximated by a truncated *Chebyshev* expansion then a minimum number of π points (or, alternatively, number of terms retained in the expansion) per wave is required. The proof is based on the observation that if the number of terms in the expansion exceeds this minimum of π per wave then the error decays *exponentially* (see [3, page 35]).

The situation changes when the wave function is discontinuous. As a generic example, consider the above mentioned function $f(x) = e^{i\pi\omega x}$ defined on $[-1, 1]$ where ω is not an integer. In this case, the truncated Fourier expansion does not converge at all in the maximum norm. This is known as the *Gibbs phenomenon* and there is no meaning to the question of number of points required to resolve such a wave.

Recently, however, the Gibbs phenomenon has been resolved. In [4] it has been shown that the first N Fourier coefficients of an analytic, but nonperiodic function contain enough information on the solution to construct an exponentially convergent *Gegenbauer* series.

Consequently we are concerned with resolution properties of the expansion in Gegenbauer polynomials $C_n^\lambda(x)$ to approximate a non-periodic wave function $f(x) = e^{i\pi\omega x}$ where ω is not an integer. We first consider the question of minimum number of points per wave needed to resolve this function. Two separate situations are considered. The first involves a fixed λ in the Gegenbauer expansion. Note that the Chebyshev expansion is a special case with $\lambda = 0$. For this situation our results show that the minimum number of points per wave is π, the same as in the special Chebyshev case. In the second situation we assume that $\gamma = \frac{\lambda}{m}$ is a constant in the Gegenbauer expansion. We show that the minimum number of points per wave increases with γ at the boundary $x = \pm 1$ (i.e. at the discontinuity of the wave) and decreases with γ at the center $x = 0$. This second situation corresponds to the case in which the first N Fourier coefficients of the function are known, since in this case we can reconstruct the Gegenbauer expansion for $\gamma = \frac{\lambda}{m}$ constant.

In Section 2 we quote some results about Gegenbauer polynomials. In Section 3 we study the resolution properties of the Gegenbauer expansion of a wave function, first for λ fixed and then for $\gamma = \frac{\lambda}{m}$ constant. Section 4 is devoted to the discussion of how many Gegenbauer coefficients can be obtained accurately from the first N Fourier coefficients. Here we have the minimum number of points per (discontinuous) wave to get exponential decay of the

error. Section 5 brings numerical confirmation for the above results.

We will use A to denote a generic constant independent of all the growing parameters throughout this paper. The actual value of A may be different at different locations.

2 Preliminaries

In this section we will collect some definitions, equalities and inequalities which will be used in later sections.

We start by defining the Gegenbauer polynomials $C_n^\lambda(x)$.

Definition 2.1 For $\lambda \geq 0$, the Gegenbauer polynomial $C_n^\lambda(x)$ is the polynomial of degree n that satisfies

$$\int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}}C_k^\lambda(x)C_n^\lambda(x)dx = 0 \qquad k \neq n \tag{2.1}$$

with

$$C_n^\lambda(1) = \frac{\Gamma(n+2\lambda)}{n!\Gamma(2\lambda)}. \tag{2.2}$$

The Gegenbauer polynomial $C_n^\lambda(x)$ achieves its maximum at the boundary $x = 1$ [2, page 206]

$$|C_n^\lambda(x)| \leq C_n^\lambda(1) \tag{2.3}$$

and its value at the center $x = 0$ is given by [1, page 777]

$$C_{2n}^\lambda(0) = (-1)^n\frac{\Gamma(\lambda+n)}{n!\Gamma(\lambda)}; \qquad C_{2n+1}^\lambda(0) = 0. \tag{2.4}$$

The Gegenbauer polynomials thus defined are not orthonormal. The norm of $C_n^\lambda(x)$ is given by [2, page 174]

$$h_n^\lambda = \int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}}C_n^\lambda(x)C_n^\lambda(x)dx = \pi^{\frac{1}{2}}C_n^\lambda(1)\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)(n+\lambda)}. \tag{2.5}$$

We will also need the following identity, which can be found in [2, page 213]

$$\frac{1}{h_l^\lambda}\int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}}e^{i\pi\omega x}C_l^\lambda(x)dx = \Gamma(\lambda)\left(\frac{2}{\pi\omega}\right)^\lambda i^l(l+\lambda)J_{l+\lambda}(\pi\omega). \tag{2.6}$$

Throughout this paper we repeatedly use the Stirling's formula: for any $x \geq 1$,

$$\Gamma(x+1) \leq (2\pi)^{\frac{1}{2}}x^{x+\frac{1}{2}}e^{-x}e^{\frac{1}{12}} \tag{2.7}$$

$$\Gamma(x+1) \geq (2\pi)^{\frac{1}{2}}x^{x+\frac{1}{2}}e^{-x} \quad . \tag{2.8}$$

We will need the following estimates for the Bessel function $J_n(nz)$ [1, page 362]

$$|J_n(nz)| \leq \left(\frac{ze^{\sqrt{1-z^2}}}{1+\sqrt{1-z^2}} \right)^n, \qquad 0 \leq z \leq 1. \tag{2.9}$$

The following Lemma will be used in conjunction with the estimate in (2.9).

Lemma 2.1

If $q(z)$ and $p(z)$ are defined by

$$q(z) = \frac{ze^{\sqrt{1-z^2}}}{1+\sqrt{1-z^2}}, \qquad p(z) = \frac{q(z)^{1+\gamma}}{z^\gamma} \tag{2.10}$$

where $\gamma \geq 0$, and the constant $c(\gamma)$ is defined by

$$c(\gamma) = \frac{\sqrt{1+2\gamma}}{1+\gamma}, \tag{2.11}$$

then

(1) For $0 \leq z < 1$, $q(z)$ is a strictly increasing function and $q(z) < 1$;

(2) For $0 < z < c(\gamma)$, $p(z)$ is a strictly increasing function;

(3) For $c(\gamma) < z < 1$, $p(z)$ a strictly decreasing function and $p(z) > 1$. □

3 Wave Resolution Properties of Gegenbauer Expansions

Consider the non-periodic wave function

$$f(x) = e^{i\pi\omega x} \tag{3.1}$$

where $\omega > 0$ is not an integer. The Gegenbauer coefficients of this function are defined by

$$\hat{f}^\lambda(l) = \frac{1}{h_l^\lambda} \int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} e^{i\pi\omega x} C_l^\lambda(x) dx \tag{3.2}$$

with h_l^λ given by (2.5).

Our objective is to find a lower bound on the ratio

$$r = \frac{m}{\omega} \tag{3.3}$$

where ω is the number of waves in (3.1) and m is the number of terms in the Gegenbauer expansion

$$f_m^\lambda(x) = \sum_{l=0}^{m} \hat{f}^\lambda(l) C_l^\lambda(x) \tag{3.4}$$

such that the approximation error using the expansion (3.4) is exponentially small when $m \to \infty$. The ratio r defined in (3.3) is usually called the number of points per wave. It is literally the number of points per wave for the collocation case and can be called number of modes per wave for the Galerkin expansion (3.4)-(3.2). We can then define the regularization error $RE(\lambda, m, r, x)$ to be

$$RE(\lambda, m, r, x) = |f(x) - f_m^\lambda(x)| \tag{3.5}$$

and ask the question of finding a lower bound $r_0 = r_0(\lambda, x)$ such that the regularization error (3.5) is exponentially small for $r > r_0$ when $m \to \infty$.

According to the identity (2.6) and the definitions (3.2)–(3.4), the regularization error (3.5) can be explicitly expressed as

$$RE(\lambda, m, r, x) = \left| \Gamma(\lambda) \left(\frac{2r}{\pi m}\right)^\lambda \sum_{l=m+1}^{\infty} i^l (l+\lambda) J_{l+\lambda}\left(\frac{\pi m}{r}\right) C_l^\lambda(x) \right| . \tag{3.6}$$

first we have the following theorem for the case with fixed λ.

Theorem 3.1 If λ is fixed, then $r_0 = \pi$ is a lower bound for the number of points per wave to obtain exponentially small regularization error for all $-1 \le x \le 1$.

Proof: Assume $r > \pi$. Denote $z_l = \frac{\pi m}{r(l+\lambda)}$ and $z = \frac{\pi}{r}$. For $l > m$ we have $z_l < z < 1$ and hence $q(z_l) < q(z) < 1$ according to Lemma 2.1. Take $s = \frac{1+q(z)}{2} < 1$ and m big enough so that $\frac{(l+1+\lambda)(l+2\lambda)}{(l+\lambda)(l+1)} q(z) \le s$ for $l > m$. If we define

$$B(l) = (l+\lambda)\,(q(z))^{l+\lambda}\,\frac{\Gamma(l+2\lambda)}{l!} \tag{3.7}$$

we have, for $l > m$

$$\frac{B(l+1)}{B(l)} = \frac{(l+1+\lambda)(l+2\lambda)}{(l+\lambda)(l+1)} q(z) \le s < 1 . \tag{3.8}$$

We can then start from the explicit formula (3.6) and absorb all the λ dependent terms into the generic constant A to obtain

$$\begin{aligned} RE(\lambda, m, r, x) &\le A m^{-\lambda} \sum_{l=m+1}^{\infty} (l+\lambda) \left| J_{l+\lambda}\left(\frac{\pi m}{r}\right) \right| C_l^\lambda(1) \\ &\le A m^{-\lambda} \sum_{l=m+1}^{\infty} (l+\lambda)\,(q(z_l))^{l+\lambda}\,\frac{\Gamma(l+2\lambda)}{l!} \end{aligned}$$

$$
\begin{aligned}
&\leq Am^{-\lambda} \sum_{l=m+1}^{\infty} B(l) \\
&\leq Am^{-\lambda} \frac{B(m)}{1-s} \\
&\leq Am^{\lambda} (q(z))^{m+\lambda}
\end{aligned}
$$

where we have used (2.3) in the first inequality (2.9) and (2.2) in the second inequality, the definition of $B(l)$ in (3.7), the fact that $q(z_l) < q(z)$ in the third inequality, and (3.8) in the fourth inequality. This finishes the proof because $q(z) < 1$. □

The result of Theorem 3.1 is not surprising; for $\lambda = 0$ this is simply the known result for the Chebyshev case [3].

For fixed λ, the estimates are essentially the same for the boundary point $x = 1$ or for the center point $x = 0$, since there is only an algebraic difference between $C_l^{\lambda}(1)$ and $C_l^{\lambda}(0)$. However, this algebraic difference becomes bigger when λ increases.

For our purpose we are more interested in the case $\lambda \sim m$. This is because we are interested in the situation where the finite *Fourier* series of $f(x)$ is given. In [4] we proved that it is possible to recover uniform exponential accuracy from the finite Fourier series of a non-periodic analytic function through the use of Gegenbauer expansions with $\lambda \sim m$.

Let us now assume that $\gamma = \frac{\lambda}{m}$ is a constant. We have then the following theorem.

Theorem 3.2 If $\gamma = \frac{\lambda}{m}$ is a constant, then

(1) The regularization error $RE(\gamma m, m, r, 1)$, at the boundary $x = 1$, is exponentially small if $r > \max(r_1, r_2)$ where r_1 is the unique solution of

$$
p\left(\frac{\pi}{r_1(1+\gamma)}\right) = \frac{(2e\gamma(1+\gamma))^{\gamma}}{(1+2\gamma)^{1+2\gamma}} \equiv Q_1(\gamma) \tag{3.9}
$$

in the region $r_1 \geq \frac{\pi}{\sqrt{1+2\gamma}}$ if (3.9) has a solution (i.e., if $p(c(\gamma)) \geq Q_1(\gamma)$), or $r_1 = \frac{\pi}{\sqrt{1+2\gamma}}$ otherwise. Also, r_2 is the unique solution of

$$
q\left(\frac{\pi}{r_2(1+\gamma)}\right) = \frac{1}{1+2\gamma}; \tag{3.10}
$$

(2) The regularization error $RE(\gamma m, m, r, 0)$, at the center $x = 0$, is exponentially small if $r > \max(r_3, r_4)$ where r_3 is the unique solution of

$$
p\left(\frac{\pi}{r_3(1+\gamma)}\right) = \frac{(e(1+\gamma))^{\gamma}}{(1+2\gamma)^{\frac{1}{2}+\gamma}} \equiv Q_3(\gamma) \tag{3.11}
$$

within the region $r_3 \geq \frac{\pi}{\sqrt{1+2\gamma}}$ if (3.11) has a solution (i.e., if $p(c(\gamma)) \geq Q_3(\gamma)$), or $r_3 = \frac{\pi}{\sqrt{1+2\gamma}}$ otherwise. Here r_4 is the unique solution of

$$q\left(\frac{\pi}{r_4(1+\gamma)}\right) = \frac{1}{\sqrt{1+2\gamma}}. \tag{3.12}$$

The functions $p(z)$ and $q(z)$ in (3.9)-(3.12) are defined by (2.10) in Lemma 2.1 and the constant $c(\gamma)$ is defined by (2.11).

Proof: Denote $z_l = \frac{\pi m}{r(l+\lambda)}$ and $z = \frac{\pi m}{r(m+\lambda)} = \frac{\pi}{r(1+\gamma)}$. For $l > m$ we have $z_l < z \leq c(\gamma)$ when $r > r_1 \geq \frac{\pi}{\sqrt{1+2\gamma}}$; hence $q(z_l) < q(z) < \frac{1}{1+2\gamma}$ according to Lemma 2.1 and the condition (3.10). Take $s = \frac{1+(1+2\gamma)q(z)}{2}$, which yields $(1+2\gamma)q(z) < s < 1$. Hence we can take m big enough so that for $l > m$,

$$\begin{aligned}\frac{(l+1+\gamma m)(l+2\gamma m)}{(l+\gamma m)(l+1)}q(z) &= \left(1+\frac{1}{l+\gamma m}\right)\left(\frac{l}{l+1}\right)\left(\frac{l+2\gamma m}{l+2\gamma l}\right)(1+2\gamma)q(z) \\ &\leq \left(1+\frac{1}{l+\gamma m}\right)(1+2\gamma)q(z) \leq s < 1\,.\end{aligned}$$

If we now define

$$B(l) = (l+\gamma m)\,(q(z))^{l+\gamma m}\,\frac{\Gamma(l+2\gamma m)}{l!}, \tag{3.13}$$

we have for $l > m$

$$\frac{B(l+1)}{B(l)} = \frac{(l+1+\gamma m)(l+2\gamma m)}{(l+\gamma m)(l+1)}q(z) \leq s < 1\,. \tag{3.14}$$

We can then start from the explicit formula (3.6) to obtain

$$\begin{aligned}RE(\gamma m, m, r, 1) &\leq \Gamma(\gamma m)\left(\frac{2r}{\pi m}\right)^{\gamma m}\sum_{l=m+1}^{\infty}(l+\gamma m)\left|J_{l+\gamma m}\left(\frac{\pi m}{r}\right)\right| C_l^{\gamma m}(1) \\ &\leq \frac{\Gamma(\gamma m)}{\Gamma(2\gamma m)}\left(\frac{2r}{\pi m}\right)^{\gamma m}\sum_{l=m+1}^{\infty}(l+\gamma m)\,(q(z_l))^{l+\gamma m}\,\frac{\Gamma(l+2\gamma m)}{l!} \\ &\leq \frac{\Gamma(\gamma m)}{\Gamma(2\gamma m)}\left(\frac{2r}{\pi m}\right)^{\gamma m}\sum_{l=m+1}^{\infty}B(l) \\ &\leq \frac{\Gamma(\gamma m)}{\Gamma(2\gamma m)}\left(\frac{2r}{\pi m}\right)^{\gamma m}\frac{B(m)}{1-s} \\ &\leq A\left(\frac{(1+2\gamma)^{1+2\gamma}}{(2e\gamma(1+\gamma))^{\gamma}}p(z)\right)^{m}\end{aligned}$$

where we have used (2.9) and (2.2) in the second inequality, the definition (3.13) and the fact $q(z_l) < q(z)$ in the third inequality, (3.14) in the fourth inequality and Stirling's formula (2.7)-(2.8) in the last inequality.

According to condition (3.9) and Lemma 2.1, the number $\frac{(1+2\gamma)^{1+2\gamma}}{(2e\gamma(1+\gamma))^{\gamma}} p\left(\frac{\pi}{r(1+\gamma)}\right)$ is strictly less than one for $r > r_1$. This finishes the proof for $x = 1$.

The proof for the center point $x = 0$ is similar and is thus omitted. □

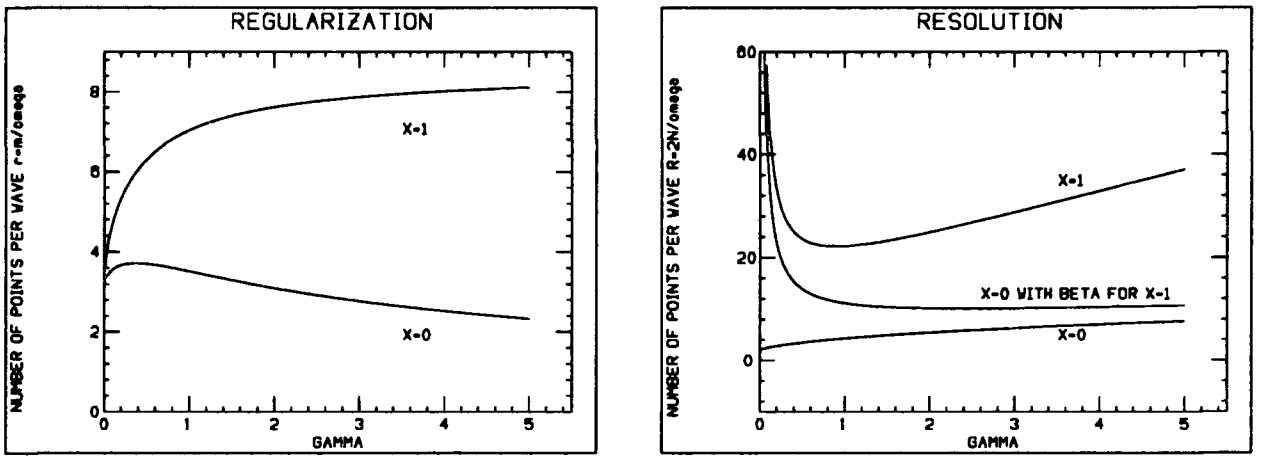

Figure 1: Minimum number of points per wave as a function of γ. Left: minimum $r = \frac{m}{\omega}$ to obtain exponentially small regularization error; Right: minimum $R = \frac{2N}{\omega}$ to obtain exponentially small resolution error from Fourier partial sum.

In Figure 1 (left) we show the curves of number of points per wave, $r = \frac{m}{\omega}$, versus γ, both for resolving the boundary $x = 1$ (upper line) and for resolving the center $x = 0$ (lower line). These were obtained from the results (3.9)-(3.12) in Theorem 3.2. We can see that the top curve for $x = 1$ is an increasing function of γ, while the bottom curve for $x = 0$ is a decreasing function of γ for $\gamma > 0.37$. At $\gamma = 1$, we need approximately 7.03 points per wave to resolve the boundary and 3.51 points per wave to resolve the center. At $\gamma = 5$, this two numbers change to 8.11 and 2.33, respectively. Figure 1 (right) is related to the truncation error to be discussed in next section.

4 The Truncation Error from Finite Fourier Series

In this section we consider the situation that $f(x) = e^{i\pi\omega x}$ is not known and only the first $2N + 1$ Fourier coefficients $\hat{f}(k)$ are given. We would like to recover, within exponential accuracy at all points in $-1 \leq x \leq 1$, the finite Gegenbauer expansion $f_m^{\lambda}(x)$ of $f(x)$ as defined by (3.4). This, together with the results in the previous section, will establish the number of points per wave for resolving discontinuous waves based on its Fourier partial sum through Gegenbauer polynomials. The result in this section parallels the general theorems in [4].

The Fourier partial sum of the discontinuous wave function $f(x) = e^{i\pi\omega x}$ is given by

$$f_N(x) = \sum_{k=-N}^{N} \hat{f}(k) e^{ik\pi x} \tag{4.1}$$

where the Fourier coefficients $\hat{f}(k)$ are defined by

$$\hat{f}(k) = \frac{1}{2}\int_{-1}^{1} f(x) e^{-ik\pi x} dx\,. \tag{4.2}$$

Assume that the first $2N+1$ Fourier coefficients $\hat{f}(k)$, $|k| \leq N$ and hence also the Fourier partial sum (4.1) are known but the function $f(x)$ is not. We thus do not know the exact Gegenbauer coefficients $\hat{f}^{\lambda}(l)$ of $f(x)$ defined by (3.2), but only the approximate ones obtained from the Fourier partial sum $f_N(x)$, which we denote by $\hat{g}^{\lambda}(l)$

$$\hat{g}^{\lambda}(l) = \frac{1}{h_l^{\lambda}} \int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} f_N(x) C_l^{\lambda}(x) dx\,. \tag{4.3}$$

Notice that $\hat{g}^{\lambda}(l)$ depends on N.

We now define the truncation error $TE(\lambda, m, N, x)$ to be

$$TE(\lambda, m, N, x) = \left| \sum_{l=0}^{m} (\hat{f}^{\lambda}(l) - \hat{g}^{\lambda}(l)) C_l^{\lambda}(x) \right|. \tag{4.4}$$

Notice that ω or $r = \frac{m}{\omega}$ is not an explicit parameter in our definition of the truncation error. The estimates we obtain later will be uniformly valid for all ω.

In the next theorem we shall bound the truncation error in terms of N, the number of given Fourier coefficients, m the number of terms in the Gegenbauer expansion, and λ.

Theorem 4.1 The truncation error at the boundary $x = 1$ satisfies the estimate

$$TE(\lambda, m, N, 1) \leq A \frac{(m+\lambda)\Gamma(m+2\lambda)\Gamma(\lambda)}{(m-1)!\Gamma(2\lambda)} \left(\frac{2}{\pi N}\right)^{\lambda-1}. \tag{4.5}$$

At the center $x = 0$, the truncation error satisfies

$$TE(\lambda, m, N, 0) \leq A \frac{(m+\lambda)\Gamma(\frac{m}{2}+\lambda)}{\Gamma\left(\frac{m}{2}\right)} \left(\frac{2}{\pi N}\right)^{\lambda-1}. \tag{4.6}$$

If $\lambda = \gamma m$ and $m = \beta N$ where γ and β are positive constants, then the truncation error satisfies

$$TE(\gamma\beta N, \beta N, N, 1) \leq A N^2 \left(\frac{\beta^{\gamma}(1+2\gamma)^{1+2\gamma}}{(2\pi e\gamma)^{\gamma}} \right)^{\beta N} \tag{4.7}$$

at the boundary $x = 1$ and

$$TE(\gamma\beta N, \beta N, N, 0) \leq AN^2 \left(\frac{\beta^\gamma (1+2\gamma)^{\frac{1}{2}+\gamma}}{(\pi e)^\gamma} \right)^{\beta N} \tag{4.8}$$

at the center $x = 0$.

Proof: Since

$$f(x) - f_N(x) = \sum_{|n|>N} \hat{f}(n) e^{in\pi x} \tag{4.9}$$

and $|\hat{f}(n)| \leq 1$ according to the definition (4.2) with $f(x) = e^{i\pi\omega x}$, we can estimate the truncation error as follows

$$\begin{aligned} TE(\lambda, m, N, x) &\leq (m+1) \max_{0\leq l\leq m} \left| (\hat{f}^\lambda(l) - \hat{g}^\lambda(l)) C_l^\lambda(x) \right| \\ &= (m+1) \max_{0\leq l\leq m} \left| \sum_{|n|>N} \hat{f}(n) C_l^\lambda(x) \left(\frac{1}{h_l^\lambda} \int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} e^{in\pi y} C_l^\lambda(y) dy \right) \right| \\ &= (m+1) \max_{0\leq l\leq m} \left| \sum_{|n|>N} \hat{f}(n) C_l^\lambda(x) \Gamma(\lambda) \left(\frac{2}{\pi n} \right)^\lambda i^l (l+\lambda) J_{l+\lambda}(\pi n) \right| \\ &\leq Am\Gamma(\lambda) \left(\frac{2}{\pi N} \right)^{\lambda-1} \max_{0\leq l\leq m} (l+\lambda) \left| C_l^\lambda(x) \right| . \end{aligned} \tag{4.10}$$

In the second step we have used (4.9), the definition of $\hat{f}^\lambda(l)$ in (3.2) and that of $\hat{g}^\lambda(l)$ in (4.3), in the third step we have used the equality (2.6), and in the last step we have used the facts $|\hat{f}(n)| \leq 1$ and $|J_\nu(x)| \leq 1$ for all x and $\nu \geq 0$ [1, page 362].

For the boundary point $x = 1$ we can then proceed as follows

$$\begin{aligned} TE(\lambda, m, N, 1) &\leq Am\Gamma(\lambda) \left(\frac{2}{\pi N} \right)^{\lambda-1} \max_{0\leq l\leq m} (l+\lambda) C_l^\lambda(1) \\ &= Am \frac{\Gamma(\lambda)}{\Gamma(2\lambda)} \left(\frac{2}{\pi N} \right)^{\lambda-1} \max_{0\leq l\leq m} \frac{(l+\lambda)\Gamma(l+2\lambda)}{l!} \\ &\leq A \frac{(m+\lambda)\Gamma(m+2\lambda)\Gamma(\lambda)}{(m-1)!\Gamma(2\lambda)} \left(\frac{2}{\pi N} \right)^{\lambda-1} \end{aligned} \tag{4.11}$$

where in the second step we have used the formula (2.2) for $C_l^\lambda(1)$, and in the last step we have used the fact that $\frac{(l+\lambda)\Gamma(l+2\lambda)}{l!}$ is an increasing function of l. The result in (4.5) is thus proven. Some simple algebra and the use of Stirling's formula (2.7)-(2.8) easily produce (4.7). The estimates (4.8) and (4.10) at the center point $x = 0$ can be obtained in a similarly fashion and the detail is thus omitted. □

Since our estimates of the truncation error, (4.7) and (4.8), do not depend on the wave number ω or $r = \frac{m}{\omega}$, the minimum number of points per wave for exponential convergence,

defined by $R = \frac{2N}{\omega}$, is obtained by using the $r = \frac{m}{\omega}$ derived in Theorem 3.2 and the largest β in (4.7) or (4.8) such that the factors on the right hand sides which are being raised to power N are still less than one. This gives the following theorem.

Theorem 4.2 Assume that the Fourier coefficients

$$\hat{f}(k) = \frac{1}{2}\int_{-1}^{1} f(x)e^{-ik\pi x}dx$$

of the function $f(x) = e^{i\omega\pi x}$ are known for $-N \leq k \leq N$. Let $\hat{g}^{\lambda}(l)$, $0 \leq l \leq m$ be the Gegenbauer expansion coefficients of $f_N(x) = \sum_{k=-N}^{N} \hat{f}(k)e^{ik\pi x}$ given by (4.3). Define the number of points per wave by

$$R = \frac{2N}{\omega} \tag{4.12}$$

and the resolution error by

$$E(\lambda, m, N, R, x) = \left| f(x) - \sum_{l=0}^{m} \hat{g}^{\lambda}(l) C_l^{\lambda}(x) \right| . \tag{4.13}$$

Then, if $\lambda = \gamma m$ and $m = \beta N$, we have the following results.

(1) At the boundary $x = 1$, if one uses

$$\beta < \frac{2\pi e\gamma}{(1+2\gamma)^{\frac{1}{\gamma}+2}} \equiv B_1(\gamma) \tag{4.14}$$

and $r = \frac{m}{\omega} > \max(r_1, r_2)$ as is derived in Theorem 3.2, then the resolution error is exponentially small and bounded as

$$E(\gamma\beta N, \beta N, N, R, 1) < A\left(N^2 b_T^N + b_R^N\right) \tag{4.15}$$

where

$$b_T = \left(\frac{\beta^{\gamma}(1+2\gamma)^{1+2\gamma}}{(2\pi e\gamma)^{\gamma}}\right)^{\beta} < 1, \qquad b_R = \left(\frac{(1+2\gamma)^{1+2\gamma}}{(2e\gamma(1+\gamma))^{\gamma}} p\left(\frac{\pi}{r(1+\gamma)}\right)\right)^{\beta} < 1 .$$

The number of points per wave $R = \frac{2N}{\omega}$ is estimated by

$$R > \frac{2\max(r_1, r_2)}{B_1(\gamma)} \tag{4.16}$$

where r_1, r_2 are defined in Theorem 3.2 and $B_1(\gamma)$ is defined by (4.14).

(2) At the center $x = 0$, if one uses

$$\beta < \frac{\pi e}{(1+2\gamma)^{\frac{1}{2\gamma}+1}} \equiv B_2(\gamma) \tag{4.17}$$

and $r = \frac{m}{\omega} > \max(r_3, r_4)$ as is derived in Theorem 3.2, then the resolution error is exponentially small

$$E(\gamma\beta N, \beta N, N, R, 0) < A\left(N^2 c_T^N + c_R^N\right) \tag{4.18}$$

where

$$c_T = \left(\frac{\beta^\gamma (1+2\gamma)^{\frac{1}{2}+\gamma}}{(\pi e)^\gamma}\right)^\beta < 1 \qquad c_R = \left(\frac{(1+2\gamma)^{\frac{1}{2}+\gamma}}{(e(1+\gamma))^\gamma} p\left(\frac{\pi}{r(1+\gamma)}\right)\right)^\beta < 1.$$

The number of points per wave $R = \frac{2N}{\omega}$ is estimated by

$$R > \frac{2\max(r_3, r_4)}{B_2(\gamma)} \tag{4.19}$$

where r_3, r_4 are defined in Theorem 3.2 and $B_2(\gamma)$ is defined by (4.17).

Proof: This is simply a combination of Theorems 3.2 and 4.1. □

In Figure 1 (right) we show the curves of number of points per wave, $R = \frac{2N}{\omega}$, versus γ, both for resolving the boundary $x = 1$ (upper line) and for resolving the center $x = 0$ (lower line), obtained from the results (4.16) and (4.19) in Theorem 4.2. We can see that the top curve for $x = 1$ achieves its minimum at around $\gamma = 0.9$. and the bottom curve for $x = 0$ is an increasing function of γ. We emphasize that these results are obtained with different β for $x = 1$ and $x = 0$ (from (4.14) and (4.17) respectively). In practice a single β should be used since one would like to resolve both the boundary and the center simultaneously. If the β for $x = 1$, given by (4.14), is also used for $x = 0$, the minimum number of points per wave, R, to resolve the center $x = 0$, would be described by the middle curve in Figure 1 (right). For the single β chosen according to (4.14), at $\gamma = 1$, a minimum of 22.2 points per wave is needed to resolve the boundary $x = 1$, and a minimum of 11.1 points per wave is needed to resolve the center $x = 0$.

5 Numerical Results and Conclusions

In this section we perform numerical calculations to demonstrate the theory developed in previous sections. We use the discontinuous wave function

$$f(x) = \cos(\pi\omega(x+1)) \tag{5.1}$$

for various wave numbers ω and report both the regularization error defined by (3.5) and the resolution error defined by (4.13).

We implement the method in the following way. The exact Gegenbauer coefficients of $f(x)$, which are needed for the regularization error, are computed using (2.6). The approximate Gegenbauer coefficients $\hat{g}^\lambda(l)$, defined in (4.3), which are needed for the resolution error, are computed using the following formula

$$\hat{g}^\lambda(l) = \delta_{0l}\hat{f}(0) + \Gamma(\lambda)i^l(l+\lambda) \sum_{0<|k|\le N} J_{l+\lambda}(\pi k)\left(\frac{2}{\pi k}\right)^\lambda \hat{f}(k) \tag{5.2}$$

where $\hat{f}(k)$ are the Fourier coefficients of $f(x)$ defined by (4.2). This formula can be easily derived from the definition of $\hat{g}^\lambda(l)$ in (4.3) and the integration formula (2.6). We compute the Bessel function $J_\nu(x)$ using an IMSL routine. The approximation to $f(x)$ is obtained by directly summing (3.4) for the regularization error, or by directly summing

$$g_m^\lambda(x) = \sum_{l=0}^{m} \hat{g}^\lambda(l) C_l^\lambda(x) \tag{5.3}$$

for the resolution error. The Gegenbauer polynomials $C_l^\lambda(x)$ are computed by the formula

$$C_l^\lambda(\cos\theta) = \sum_{k=0}^{l} \frac{\Gamma(k+\lambda)}{k!\Gamma(\lambda)} \frac{\Gamma(l-k+\lambda)}{(l-k)!\Gamma(\lambda)} \cos(l-2k)\theta \tag{5.4}$$

which can be found in [1, page 175].

We remark that the implementation techniques described above are subject to roundoff effects for large λ and m. A better way of implementing the method through Chebyshev polynomials is currently under investigation.

In Figure 2 we show the errors, in a logarithm scale, at the discontinuity $x = 1$ for $\omega = 1.4$, 2.4, 3.4 and 4.4. We choose $\gamma = \frac{\lambda}{m} = 1$ in this illustration. On the left, the regularization error $RE(m, m, \frac{m}{\omega}, 1)$ is shown as a function of m, the number of terms retained in the Gengenbauer expansion. On the right, the resolution error $E(\frac{N}{2}, \frac{N}{2}, N, \frac{2N}{\omega}, 1)$ is shown as a function of $2N$, to total the number of terms in the Fourier expansion. Here we take $\beta = 0.5$ which satisfies (4.14). We can see that the errors are order $O(1)$ until $r = \frac{m}{\omega}$ or $R = \frac{2N}{\omega}$ reaches the critical values obtained in Theorem 3.2 and in Theorem 4.2, after which the errors drop exponentially. Figure 3 shows the same result but for the center point $x = 0$.

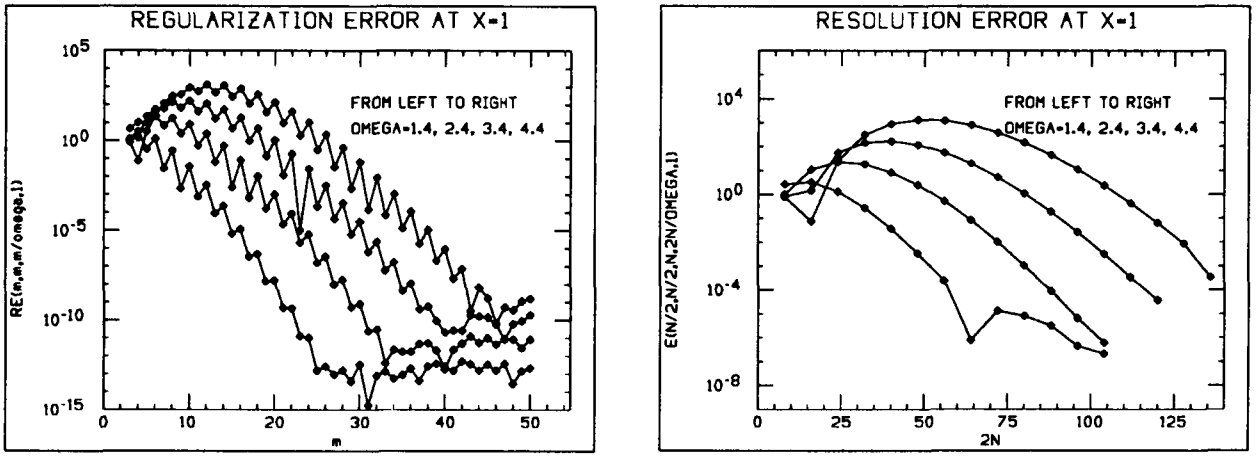

Figure 2: Logarithmic error at the discontinuity $x = 1$ for (5.1) with $\omega = 1.4$, 2.4, 3.4 and 4.4. Here $\gamma = \frac{\lambda}{m} = 1$. Left: the regularization error $RE(m, m, \frac{m}{\omega}, 1)$ as a function of m; Right: the resolution error $E(\frac{N}{2}, \frac{N}{2}, N, \frac{2N}{\omega}, 1)$ as a function of $2N$, $\beta = 0.5$.

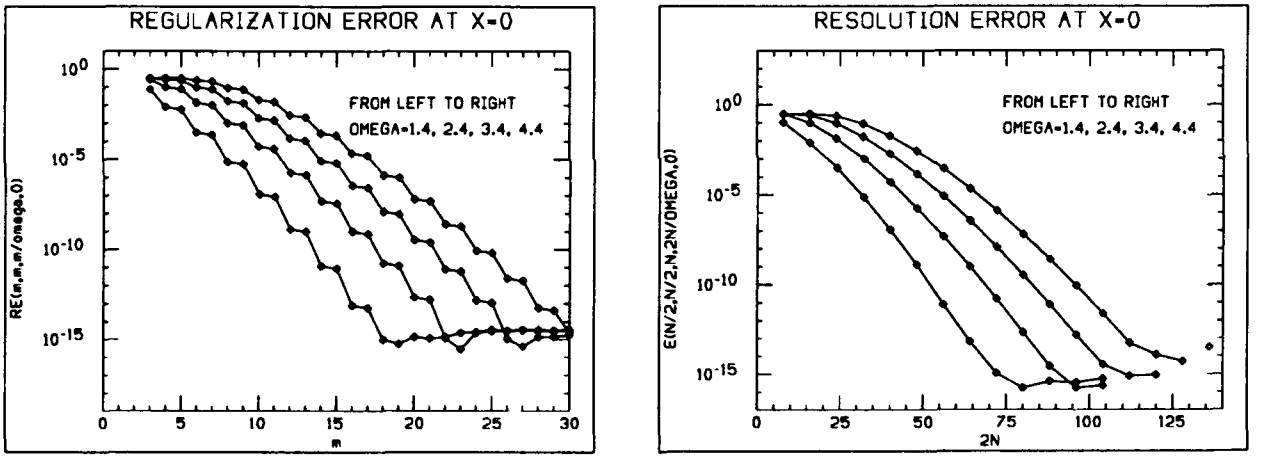

Figure 3: Same as Figure 2 but for the center point $x = 0$.

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In this paper we discuss the wave-resolution properties of the Fourier approximations of a wave function with discontinuities. It is well known that a minimum of two points per wave is needed to resolve a periodic wave function using Fourier expansions. For Chebyshev approximations of a wave function, a minimum of π points per wave is needed (3). Here we obtain an estimate for the minimum number of points per wave to resolve a discontinuous wave based on its Fourier coefficients.

In our recent work on overcoming the Gibbs phenomenon, we have shown that the Fourier coefficients of a discontinuous function contain enough information to reconstruct with exponential accuracy the coefficients of a rapidly converging Gegenbauer expansion. We therefore study the resolution properties of a Gegenbauer expansion where both the number of terms and the order increase.

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